Flow of a Thin Liquid Film on a Wavy Wall

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The flow of a thin liquid film is one in which both surface tension and viscosity play vital roles. Thin films with two free surfaces such as in foams have been extensively studied. Analyses for film flow on a solid wall (Mysels et al., 1959; Levich, 1962; Ruschak, 1978) have been mostly limited to those on a wall of negligible curvature such as flows down a vertical or inclined wall. But a thin film flow on a curved wall whose curvature is not negligible is not unusual. It may also be that the wall is of an irregular shape, possibly with cracks and crevices. Such flows, for example, are believed to exist in the alveolar liquid lining of the human lung (Wilson, 1979). The usual assumption of small surface slopes in calculating the surface pressure does not, in general, hold good in this case. The analysis presented here is an attempt to study, and to obtain a criterion for the existence of a thin film flow on such a wavy wall.

FLOW ANALYSIS

Consider steady flow in a thin liquid film on a wavy wall whose surface is given by $Y = Y(X)$, where the X and Y coordinates are as shown in Figure 1. The x-coordinate is along the wall, in the direction of flow, and the y-coordinate perpendicular to it. The film thickness is considered to be small compared to the dimensions of the wall.

The surfactants adsorbed at the free surface are assumed not to react with or dissolve in the liquid. Based on Gibbs' analysis (1931) of such films, the film surface is assumed to have a constant speed u_s.

The flow is assumed to be two-dimensional, in the xy plane, and the velocities are small. For a steady flow, it can be shown that the velocity in the fluid film

$$u = -\frac{1}{2\mu}\frac{dp}{dx}(h-y)y + u_s\frac{y}{h}, \tag{1}$$

where p includes the gravitational potential.
The corresponding flow rate per unit width in the z-direction

$$Q = -\frac{h^3}{12\mu}\frac{dp}{dx} + \frac{u_s h}{2} \tag{2}$$

Neglecting the interfacial curvature relative to the wavy wall, the pressure gradient

$$\frac{dp}{dx} = -\rho g_x - T\frac{d}{dx}\left\{\frac{d^2Y/dX^2}{[1+(dY/dX)^2]^{3/2}}\right\},$$

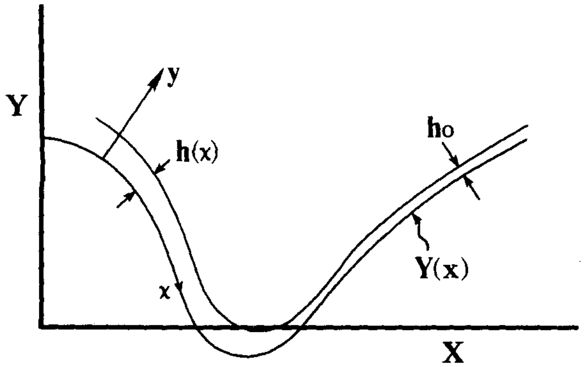

Figure 1. Film flow on a wavy wall.

where T is the surface tension, and g_x the component of g in the direction of flow. g_x is considered positive for flow down a wall. $(d^2Y/dX^2)/[1+(dY/dX)^2]^{3/2}$ represents the curvature of the wavy wall. Equation 2 in this case becomes

$$Q = \frac{h^3}{12\mu}\left[\rho g_x + T\frac{d}{dx}\left\{\frac{d^2Y/dX^2}{[1+(dY/dX)^2]^{3/2}}\right\}\right] + \frac{u_s h}{2} \tag{3}$$

An equation for the steady state film thickness $h(x)$ is obtained by considering the conservation of the flow rate Q. Consider a region of film flow where h is a constant ho, and the wall curvature is constant or small. Then from Eq. 3,

$$Q = \frac{\rho g_x h^3}{12\mu} + \frac{h^3 T}{12\mu}\frac{d}{dx}\left\{\frac{d^2Y/dX^2}{[1+(dY/dX)^2]^{3/2}}\right\} + \frac{u_s h}{2} = \frac{\rho g_{xo} ho^3}{12\mu} + \frac{u_s ho}{2}, \tag{4}$$

where g_{xo} is the x-component of g in the region of uniform film thickness. Introducing the dimensionless variables

$\eta = h/ho$, $\xi = (6\mu u_s T)^{1/2}X/ho$, $\zeta = (6\mu u_s/T)^{1/2}x/ho$,

$\alpha = \left(\frac{6\mu u_s}{T}\right)^{1/2} Y/ho$, $G = \rho g_x ho^2/(6\mu u_s)$, and

$Go = \rho g_{xo} ho^2/(6\mu u_s)$, Eq. 4 becomes

$$\frac{d}{d\zeta}\left\{\frac{d^2\alpha/d\xi^2}{[1+(d\alpha/d\xi)^2]^{3/2}}\right\}\eta^3 + G\eta^3 + \eta - (1+Go) = 0 \tag{5}$$

If the dimensionless wall curvature

$$\frac{d^2\alpha/d\xi^2}{[1+(d\alpha/d\xi)^2]^{3/2}}$$

is denoted by C, Eq. 5 reduces to

$$\frac{dC}{d\zeta}\eta^3 + G\eta^3 + \eta - (1+Go) = 0$$

Or,

$$\frac{dC}{d\zeta} = -G - \frac{1}{\eta^2} + \frac{1+Go}{\eta^3},$$

where $dC/d\zeta$ represents the rate of change of the curvature of the wavy wall in the downstream direction. G and Go have the same signs as those of g_x and g_{xo} respectively.

Figure 2 gives a plot of $dC/d\zeta$ as a function of η. A solution consistent with the assumption that the film is thin exists for values of

$$\frac{dC}{d\zeta} > -\left[\frac{0.15}{(1+Go)^2} + G\right]. \tag{6}$$

If

$$dC/d\zeta < -\left[\frac{0.15}{(1+Go)^2} + G\right],$$

a thin film solution does not exist, and the valleys would presumably fill with fluid to form a pool.

For a thin film flow in which the effect of gravity is predominant, the magnitudes of G and Go are large compared to 1, and

$$\frac{dC}{d\zeta} > -G. \tag{7}$$

For flow in a very thin film (h and ho very small), such as in the

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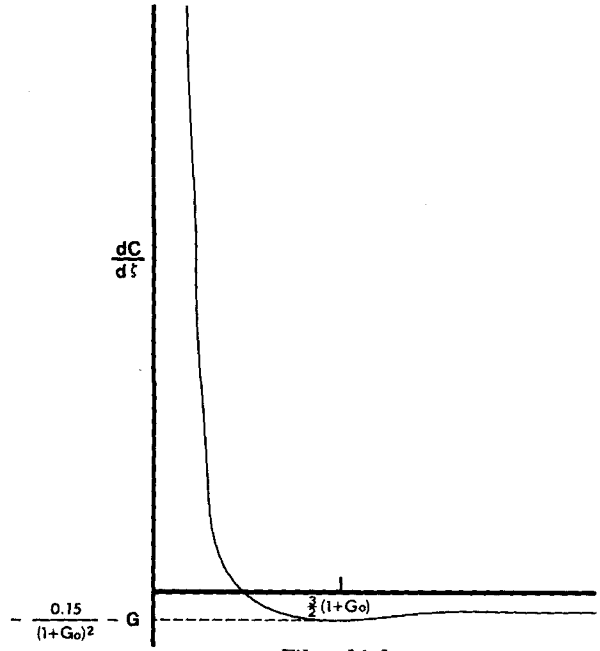

Figure 2. Rate of change of wall curvature as a function of film thickness.

alveolar liquid lining of the human lung, the effect of gravity is insignificant, $|G|, |Go| \ll 1$, and

$$\frac{dC}{d\zeta} > -0.15. \qquad (8)$$

(For example, if $\rho g_x \sim 10^3$ dyne/cm^3, $ho \sim 1\ \mu = 10^{-4}$ cm, $\mu \sim$ 10 cp, $u_s \sim 0.1$ cm/s, then $G \sim 10^{-4}$.)

DISCUSSION

Consider the wavy wall shown in Figure 1. For the region where the flow is downward, G is positive, and the rate of change of wall curvature in the direction of flow is apparently positive. This is in conformity with the conditions 6, 7 and 8, and a thin film flow is possible in this region. For the region where the wall curves upward, G is negative, and the rate of change of wall curvature is apparently negative. A thin film flow governed by relation 7 is impossible in this region. Also a very thin film flow is possible only if the rate of change of wall curvature satisfies condition 8.

In general, it may be concluded that it is the rate of change of wall curvature in the direction of flow rather than the curvature itself that is the criterion for the existence of a thin film flow on a wavy wall.

ACKNOWLEDGMENT

The author wishes to thank T. A. Wilson (University of Minnesota) for the many useful discussions.

NOTATION

C = film surface curvature, dimensionless
g = acceleration due to gravity
g_x = component of g in the direction of flow
g_{xo} = component of g in the direction of flow in the region of uniform film thickness
G = gravity constant, $\frac{\rho g_x ho^2}{6\mu u_s}$
Go = gravity constant, $\frac{\rho g_{xo} ho^2}{6\mu u_s}$
ho = uniform film thickness
p = pressure
Q = flow rate per unit width
T = surface tension
u = fluid velocity in the film
u_s = film surface velocity
x,y = axes of coordinates for the film
X,Y = axes of coordinates for the wavy wall
z = coordinate perpendicular to the plane of flow

Greek Letters

α = height of wavy wall, $\left(\frac{6\mu u_s}{T}\right)^{1/2} Y/ho$, dimensionless
ζ = distance, $\left(\frac{6\mu u_s}{T}\right)^{1/2} x/ho$, dimensionless
η = film thickness, h/ho, dimensionless
μ = viscosity
ξ = distance, $\left(\frac{6\mu u_s}{T}\right)^{1/2} X/ho$, dimensionless
ρ = fuid density

LITERATURE CITED

Gibbs, J. W., *Collected Works*, **I**, 301, Longmans Green, New York (1931).
Levich, V. G., *Physicochemical Hydrodynamics*, 669, Prentice-Hall, Englewood Cliffs, NJ (1962).
Mysels, K. J., K. Shinoda, and S. Franklin, *Soap Films—Studies of Their Thinning*, Pergamon Press, New York (1959).
Ruschak, K. J., "Flow of a Falling Film into a Pool," *AIChE J.*, **24**, 705 (1978).
Wilson, T. A., "Paranchymal Mechanics at the Alveolar Level," *Proc. Am. Soc. Exp. Biol.*, **38**, 7 (1979).

Manuscript received April 12, 1982; revision received June 28, and accepted July 1, 1983.